WINNING GAMES FOR BOUNDED GEODESICS IN TEICHMÜLLER DISCS

JONATHAN CHAIKA, YITWAH CHEUNG, HOWARD MASUR

ABSTRACT. We prove that for the flat surface defined by a holomorphic quadratic differential the set of directions such that the corresponding Teichmüller geodesic lies in a compact set in the corresponding stratum is a winning set in Schmidt game. This generalizes a classical result in the case of the torus due to Schmidt and strengthens a result of Kleinbock and Weiss.

1. STATEMENT OF THEOREM

The purpose of this paper is to prove the following theorem.

Theorem 1. *Let q be a holomorphic quadratic differential on a closed Riemann surface of genus $g > 1$. Then the set of directions θ in the circle S^1 such that the Teichmüller geodesic defined by $e^{i\theta}q$ stays in a compact set of the corresponding stratum in the moduli space of quadratic differentials is a winning set for Schmidt game and absolute winning in the sense of McMullen.*

As an immediate corollary we get the following result which was first proved by Kleinbock and Weiss [2] using quantitative non-divergence of horocycles [4].

Corollary 1. The set of directions such that the Teichmüller geodesic stays in a bounded set has Hausdorff dimension 1.

1.1. **Acknowledgments.** The authors would like to thank Dmitry Kleinbock and Barak Weiss for telling the first author about this delightful problem and helpful conversations.

2. INTRODUCTION

2.1. **Schmidt games.** We describe the Schmidt game in $\mathbb{R}^n$. Suppose we are given a set $E \subset \mathbb{R}^n$. Suppose two players Bob and Alice take turns choosing a sequence of closed Euclidean balls

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset B_3 \dots$$

(Bob choosing the B_i and Alice the A_i) whose diameters satisfy, for fixed $0 < \alpha, \beta < 1$, $|A_i| = \alpha|B_i|$ and $|B_{i+1}| = \beta|A_i|$. Following Schmidt

Definition 1. *We say E is an (α, β) winning set if Alice has a strategy so that no matter what Bob does, $\cap_{i=1}^{\infty} B_i \subset E$. We say E is α-winning if it is (α, β) winning for all $0 < \beta < 1$ and winning if it is α winning for some α.*

One important consequence of winning sets is they have full Hausdorff dimension. McMullen [3] suggested an improvement of the Schmidt game as follows. Bob and Alice in sequence choose a sequence of balls B_i, A_i such that the sets

$$B_1 \supset B_1 \setminus A_1 \supset B_2 \supset B_2 \setminus A_2 \supset B_3 \dots$$

are nested and for fixed $0 < \beta < 1/3$, $|B_{i+1}| \geq \beta |B_i|$ and $|A_i| \leq \beta |B_i|$

Definition 2. *We say E is a β-absolute winning set if Alice has a strategy so that $\cap_{i=1}^{\infty} B_i \cap E \neq \emptyset$.*

Definition 3. *We say E is absolute winning set if for all $0 < \beta < 1/3$, E is β-absolute winning.*

As McMullen shows, absolute winning implies that the countable intersection of their images under bi-lipshitz maps are non-empty. Unlike winning this holds even if the Lipshitz constants go to zero [3]. McMullen also showed that unlike winning in the sense of Schmidt absolute winning sets are invariant under quasisymmetric maps. He also showed absolute winning implies Schmidt winning. In the same paper McMullen provided an example of a set which is Schmidt winning but not absolute winning.

2.2. **Outline.** The theorem in the case of the torus is equivalent to the statement that there is a strategy such that the point $y = \cap_{i=1}^{\infty} B_i$ satisfies

$$\inf_{\frac{p}{q} \in \mathbb{Q}} q^2 |\frac{p}{q} - y| > 0.$$

The key idea to finding a strategy so that y satisfies the above inequality begins with the simple observation that distinct rationals are *separated*; that is, $|\frac{p_1}{q_1} - \frac{p_2}{q_2}| \geq \frac{1}{q_1 q_2}$. In geometric terms this is the statement that two intersecting closed curves cannot be simultaneously short on any torus. This allows one to conclude that in any interval B_j, chosen by Bob, there is at most one fraction $\frac{p}{q}$ whose denominator is bounded in terms in terms of $|B_j|$. Then Alice chooses her interval to contain that fraction, blocking it in the sense that $q^2 |\frac{p}{q} - y|$ is bounded below for any y in the complement.

In the game played with quadratic differentials on a higher genus we have the same criterion for a Teichmüller geodesic to lie in a compact set as the above criterion for fractions. This is given in Proposition 1. This says that in order to show this set is winning we want to find a strategy giving us a point far from the direction of any saddle connection. Unlike the genus 1 case we have the major complication that directions of saddle connections in general need not be separated. It may happen that on some flat surfaces there are many intersecting short saddle connections. This forces us to consider *complexes* of saddle connections that become simultaneously short under the flow. We call these complexes *shrinkable*.

The process of combining a pair of shrinkable complexes of a certain level or complexity to build a shrinkable complex of higher level is given by Lemma 7 with the preliminary Lemma 2. These ideas are not really new; having appeared in several papers beginning in [1].

The main point in this paper and the strategy is given by Theorem 2. We show first that complexes of highest level are separated as in the case of the torus for otherwise we could build a complex of higher level which is shrinkable by combining them. We develop a strategy for Alice where, as in the torus case, she blocks

these highest level complexes. Then we consider complexes of one lower level which essentially lie in the complementary interval and which are not too long in a certain sense. We show these are separated as well, for if not, we could combine them into a highest level complex of bounded size and these have been blocked at the previous stage. Thus there can be at most one such lower level complex and we block it. We continue this process inductively considering complexes of decreaing level one step at a time, ending by blocking single saddle connections. Then after a fixed number of steps we return by blocking highest level complexes and so forth. From this strategy the main theorem will follow.

2.3. **Quadratic differentials.** We will denote $q = \phi(z)dz^2$ as a holomorphic quadratic differential on a Riemann surface X of genus $g > 1$. We will denote by (X, q) the corresponding flat surface. Suppose q has zeroes of orders $k_1, \ldots, k_p$ with $\sum k_i = 4g - 4$. There is a moduli space or stratum $\mathcal{Q} = \mathcal{Q}(k_1, \ldots, k_p, \pm)$ of quadratic differentials all of which have zeroes of orders k_i. The $+$ sign occurs if q is the square of an Abelian differential and the $-$ sign otherwise.

A quadratic differential q defines an area form $|\phi(z)||dz^2|$ and a metric $|\phi|^{1/2}|dz|$. We assume that our quadratic differentials have area 1. Recall a *saddle connection* is a geodesic in the metric joining a pair of zeroes which has no zeroes in its interior.

A choice of a branch of $\phi^{1/2}(z)$ along a saddle connection β and an orientation of β determines a holonomy vector

$$hol(\beta) = \int_\beta \phi^{1/2}dz \in \mathbb{C}.$$

It is defined up to $\pm$. Thinking of this as a vector in $\mathbb{R}^2$ gives us the horizontal and vertical components defined up to $\pm$. We will denote by $h(\gamma)$ and $v(\gamma)$ the absolute value of these components. We will denote its length $\gamma|$ as the maximum of $h(\gamma)$ and $v(\gamma)$. This slightly different definition will cause no difficulties in the sequel.

Given $\epsilon > 0$, let $\mathcal{Q}_\epsilon$ denote the set of unit area quadratic differentials in the stratum such that the shortest saddle connection has length at least ϵ. The group $SL(2, \mathbb{R})$ acts on $\mathcal{Q}$. (In the action we will suppress the underlying Riemann surface). Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

denote the Teichmüller flow acting on $\mathcal{Q}$ and

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

denote the rotation subgroup.

The Teichmuller flow acts by expanding the horizontal component of saddle connections by a factor of e^t and contracting the vertical components by e^t.

Definition 4. *We say a direction θ is bounded if there exists ϵ such that $g_t r_\theta q \in \mathcal{Q}_\epsilon$ for all $t \geq 0$.*

2.4. **Conditions for β absolute winning.**

Definition 5. *Given a saddle connection γ on (X, q) we denote by θ_γ the direction that makes it vertical.*

We can think of the set of saddle connections as a subset of $S^1 \times \mathbb{R}$ by associating to each γ the pair $(\theta_\gamma, |\gamma|)$.

We first show that bounded trajectories are the trajectories which avoid saddle connections. This observation will be the motivation for what follows.

Proposition 1. *Let $S = \{(\theta, L) : \theta$ is the direction of a saddle connection of length $L\}$. If $\inf\limits_{(\theta,L) \in S} L^2 |\theta - \psi| > 0$, then ψ determines a bounded direction.*

Proof. If $(\theta, L) \in S$ and $|\theta - \psi| > c$, for a constant c, then the length of the saddle connection in $g_t r_\psi q$ coming from (θ, L) is at least $\max\{e^t \sin(c)L, e^{-t}L\}$. This is minimized when equality of the two terms holds; that is when when $t = \frac{-\log(\sin(c))}{2} = -\log(\sqrt{\sin(c)})$. So if $c > \frac{\epsilon}{L^2}$, where $\epsilon < 1$, then $\max\{e^t \sin(c)L, e^{-t}L\} > \epsilon^{1/2}$. $\qquad\square$

2.5. Complexes.

Definition 6. *We say two saddle connections of (X, q) are disjoint if they meet at most at their endpoints.*

Definition 7. *A complex of (X, q) is a union of disjoint saddle connections. If every edge of a complex K on a surface (X, q) has length at most ϵ then we say K is an ϵ-complex.*

We adopt the convention that if saddle connections bound a triangle, then we add the triangle to the complex. This allows for a complex to have an interior and allows discussion of the boundary of a complex.

Definition 8. *The level of a complex is the number of saddle connections in the complex.*

An Euler characteristic argument says that the number of saddle connections needed to triangulate the surface is $6g - 6 + 3p$. Recall that p is the number of singularities.

Lemma 1. *There exists ϵ such that a 3ϵ-complex must have fewer than $6g - 6 + 3p$ saddle connections.*

Proof. Continually adding disjoint saddle connections forces a triangulation of the surface. If the surface can be triangulated by edges of length ϵ then there is a bound in terms of ϵ for the area. However we are assuming that the area of (X, q) is 1. $\quad\square$

Definition 9. *Given q, there is a number M which is the maximum level of any ϵ-complex for any $g_t r_\theta q$.*

Definition 10. *Two complexes K_1, K_2 are combinatorially equivalent if they have the same level and the same boundary saddle connections and their union is not the whoe surface.*

The following lemma is the basic geometric construction.

Lemma 2. *Suppose K is an ϵ-complexes of level i on some (X, q) and γ is a saddle connection of length at most ϵ that does not lie in the closure of K Then there is a complex $\tilde{K}$ of level $i + 1$ on (X, q) formed by adding a new saddle connection σ to K such that $|\sigma| \leq 3 \max_{\gamma_1 \in K_1 \cup \{\gamma\}} |\gamma_1|$.*

Proof. We have that γ must be either disjoint from K or cross the boundary of K.

Case I. γ is disjoint from K. Add γ to K to form $\tilde{K}$. It is clear that the conclusion holds.

Case II γ intersects ∂K crossing $\beta \subset \partial K$ at a point p dividing β into segments β_1, β_2. Let β_1 maximize $v(\partial K_1)$.

Case IIa One endpoint p' of γ lies in the exterior of K_1.

Let $\hat{\gamma}$ be the segment of γ that goes from p' to p. We consider the homotopy class of paths $\hat{\gamma} * \beta_i$. Together with β they bound a simply connected domain Δ. Replace each path by the geodesic ω_i joining the endpoints in the homotopy class. Then $\partial \Delta$ is made up of at most M saddle connections denoted γ' all of which have their horizontal and vertical lengths bounded by the sum of the horizontal and vertical lengths of γ and β. If some $\gamma' \notin K$ we add it to form $\tilde{K}$. It is clear that the conclusion holds.

The other possibility is that $\Delta \subset \partial K$. We cannot have that Δ is a triangle since then Δ would be a subset of K, contradicting the assumption on γ. Since the edges of Δ all have length at most ϵ we can find a diagonal in Δ of length at most 2ϵ and add it to form $\tilde{K}$. In this case $\partial \tilde{K} = \partial K$ so the conclusion holds.

Case IIb Both endpoints of γ lie in K. Let γ successively cross ∂K at p_1, p_2, and let γ_1 be the segment of γ lying in the exterior of K between p_1 and p_2.

The first case is where p_1, p_2 lie on different β_1, β_2. They divide β_i into segments β_i', β_i''. We can form a pair of homotopy classes $\beta_1' * \gamma_1 * \beta_2'$ and $\beta_1'' * \gamma_1 * \beta_2''$. We replace these with their geodesics and then together with β_1, β_2 they bound a simply connected domain. We are then in a situation similar to Case IIa.

The last case is that p_1, p_2 lie on the same saddle connections β of ∂K. Let $\hat{\beta}$ be the segment between p_1 and p_2. let β_1 and β_2 the segments joining the endpoints of β to p_1, p_2. The loop $\gamma_1 * \hat{\beta}$, the segment β and the arc in the class of $\beta_1 * \gamma_1 * \beta_2$ bound an annulus. The analysis is similar to the previous cases. $\square$

Definition 11. *We say two ϵ complexes K_1, K_2 of level i are topologically combinable if there exists a saddle connection $\gamma_2 \subset \partial K_2$ that intersects the exterior of K_1 and a saddle connection $\gamma_1 \subset \partial K_1$ that intersects the exterior of K_2.*

Lemma 3. *There is a constant N_0 just depending on the topology of the surface such that if $K_1, K_2, \ldots K_{N_0+1}$ are combinatorially distinct ϵ complexes of level i, then there exists K_n, K_m that are topologically combinable.*

Proof. Suppose K_n, K_m are not combinable because no component of ∂K_n intersects the exterior of K_m; that is, every component of ∂K_n is either contained in the interior of K_m or is contained in ∂K_m. This implies that every component of $X \setminus K_m$ either coincides with, is contained in, or is disjoint from every component of $X \setminus K_n$. Thus given a collection of combinatorial distinct complexes no two of which are combinable there would be a collection of subsurfaces of X such that any two are disjoint or nested. Each complementary region contains a triangle that does not intersect any other complementary domain. The number of complementary domains therefore is bounded by the number of triangles needed to triangulate the surface. We take N_0 to be this number. $\square$

Now fix the base surface (X_0, q).

Remark 1. We have also fixed a constant β and an interval I_{-1} chosen by player Bob. For technical reasons Alice's first move blocks a neighborhood of the lefthand

side of I_{-1} and her second move blocks a neighborhood of the righthand side of I_0. Thus we assume I_1 has been chosen by Bob and we consider this the first move.

All angles, lengths will be measured on the base surface. Now let K be a level m complex.

Definition 12. *Denote by $L(K)$ the length of the longest saddle connection in K. Let $\theta(K)$ the angle that makes the longest saddle connection vertical.*

We assume that the complexes considered now have the property that for any saddle connection $\gamma \in K$ we have $|\theta_\gamma - \theta(K)| \leq \frac{\pi}{4}$. This implies that measured with respect to the angle $\theta(K)$ we have $|\gamma| = v(\gamma)$. In other words the vertical component is larger than the horizontal component. This will exclude at most finitely many complexes from our game and these will be excluded in any case by our choice of c_M in Theorem 2.

Definition 13. *We say a complex K is τ shrinkable if for any saddle connection β of K we have $|\theta_\beta - \theta(K)| \leq \frac{\tau^2}{|\beta|L(K)}$.*

We note that this condition could equally well be stated as follows. Let $h(\beta)$ be the horizontal component the holonomy vector of β makes with the direction $\theta(K)$. Then K is τ shrinkable if for all β, $h(\beta) \leq \frac{\tau^2}{L(K)}$.

Definition 14. *A complex K and a saddle connection γ not contained in the closure of K are jointly τ shrinkable if K is shrinkable and*

- *if $|\gamma| \leq L(K)$ then $|\theta(K) - \theta_\gamma| \leq \frac{\tau^2}{|\gamma|L(K)}$.*
- *if $L(K) < |\gamma|$ then $|\theta_\gamma - \theta_\omega| \leq \frac{\tau^2}{|\gamma||\omega|}$ for all $\omega \in K$.*

The following is immediate.

Lemma 4. *If $\tau_1 < \tau_2$ and K is τ_1 shrinkable, then it is τ_2 shinkable.*

The point of this definition is the following lemma.

Lemma 5. *Suppose K and γ are jointly τ shrinkable. Set $e^t = \frac{\max(L(K),|\gamma|)}{\tau}$. If $L(K) \geq |\gamma|$, set $\theta = \theta(K)$ and if $|\gamma| > L(K)$, set $\theta = \theta_\gamma$. Then on the surface $g_t r_\theta q$ every saddle connection of K and γ have length at most τ.*

Proof. Suppose the first possibility holds. For any saddle connection ω of K (resp. γ) its horizontal component $h(\omega)$ (resp. $h(\gamma)$) with respect to the angle $\theta(K)$ satisfies $h(\omega) \leq \frac{\tau^2}{L(K)}$. (resp. $h(\gamma) \leq \frac{\tau^2}{L(K)}$). That means that on the surface $g_t r_\theta q$ both its horizontal and vertical components are at most τ.

If the second possibility holds then we use the angle θ_γ and now horizontal lengths are bounded by $\frac{\tau^2}{|\gamma|}$ and the same analysis holds.

$\square$

The following is a converse to Lemma 5.

Lemma 6. *If there is θ, t such that every saddle connection of $g_t r_\theta q$ has length at most $\frac{\tau^2}{2}$, then K is τ shrinkable.*

Proof. Let $h(\gamma) = |\gamma||\theta - \theta_\gamma|$ the horizontal component a saddle connection γ of K makes with θ. Since the vertical component of the longest saddle must have length

at most $\frac{\tau}{2}$ on $g_t r_\theta q$, we have $e^t \geq \frac{2L(K)}{\tau}$. Then since $e^t h(\gamma) \leq \frac{\tau}{2}$, we conclude that

$$|\theta - \theta_\gamma| \leq \frac{\tau^2}{4|\gamma|L(K)},$$

and so by the triangle inequality

$$|\theta_\gamma - \theta(K)| \leq \frac{\tau^2}{2L(K)|\gamma|}.$$

$\square$

Lemma 7. *Suppose K is a i complex and K and γ are jointly τ shrinkable. Then there exists an $i + 1$ complex $\tilde{K}$ found by adding a new saddle connection σ such that*

  (1) $L(\tilde{K}) \leq 3 \max(L(K), |\gamma|)$.
  (2) $\tilde{K}$ *is 6τ shrinkable*
  (3) $|\theta_\sigma - \theta(K)| \leq \frac{4\tau^2}{L(K)|\sigma|}$.

Proof. We use Lemma 5 to find a flat surface $g_t r_\theta q$ where $e^t = \frac{\max(L(K), |\gamma|)}{\tau}$ on which K is a τ complex and $|\gamma| \leq \tau$. We apply Lemma 2 to find a 3τ complex $\tilde{K}$ by adding a new saddle connection σ. Since on that surface $|\sigma| \leq 3 \max(L(K), |\gamma|)$ flowing back to the original surface (X, q) we get the first conclusion. Since $L(\tilde{K}) \leq 3\tau$, $\tilde{K}$ is is 6τ shrinkable by Lemma 6.

On the surface $g_t r_\theta q$ we have $h(\sigma) \leq 3\tau$, This implies

$$|\theta_\sigma - \theta| \leq \frac{e^{-t}h(\sigma)}{|\sigma|} \leq \frac{3e^{-t}\tau}{|\sigma|}.$$

If $|\gamma| \leq L(K)$ so that $\theta = \theta(K)$, the third conclusion is immediate from the definition of e^t. If $L(K) < |\gamma|$ then $\theta = \theta_\gamma$ and the third conclusion follows from the definition of e^t, the fact that $|\theta_\gamma - \theta(K)| \leq \frac{\tau^2}{L(K)|\gamma|}$ and the triangle inequality. $\square$

In what follows we will be considering shrinkable complexes. In each combinatorial equivalence class of such shrinkable complexes we will consider the complex K which minimizes $L(K)$ and the corresponding angle $\theta(K)$. We note that this complex is perhaps not unique and so there is ambiguity in $\theta(K)$ but this will not matter.

The statement and proof of the next theorem are fairly technical and long so we give an outline of the main ideas first. In our game Bob presents Alice with an initial interval. We will first consider shrinkable complexes of maximal level M of a certain bounded size, whose angles are close to the given interval. We show that no two of them be combinable. Otherwise they could be combined to form a bigger shrinkable complex which does not exist since M was the maximum level. We think of these complexes as separated. By Lemma 3 there are at most N_0 of these. In N_0 successive steps then we block this set of at most N_0 maximal level complexes. Then we proceed to level $M - 1$ complexes whose angles are close to the new interval given by the game after these N_0 steps, and show that there is no pair of combinable complexes; for otherwise we could combine them to form a level M complex which however was blocked in one of the previous N_0 steps. Then in the next N_0 steps we block all possible level $M - 1$ complexes. We repeat this procedure and finish by considering level 1 complexes; that is, saddle connections. After these

are blocked we then begin again after a total of $N_0 M$ steps with maximal level complexes of a slightly longer size, (roughly $\beta^{-\frac{N_0 M}{2}}$) and the game continues.

We set up the following notation in the theorem that follows. Given $j \geq 1$ let $j' = j'(j)$ be determined by

(1) $$j - N_0 + 1 \leq j' \leq j \text{ and } j' \equiv 1 \bmod N_0$$

Then for each j, let $i = i(j) \in \{1, \ldots, M\}$ be determined by

(2) $$i + [\frac{j-1}{N_0}] = i + \frac{j'-1}{N_0} = 0 \bmod M.$$

Theorem 2. *Given β, and Bob's first move I_1 in the game, there exist positive constants c_i, $i = 1, \ldots, M+1$, and a strategy for Alice such that regardless of the choices I_j made by Bob, the following will hold. For all $c_i \beta^{3+N_0 M}$ shrinkable level i-complexes K either*

(a) $|\gamma_b| \cdot |L(K)| \cdot |I_{j'}| \geq c_i^2$ *or*
(b) $d(\theta(K), I_{j'}) > \frac{\beta^{N_0 M} c_i^2}{4|\gamma_b| \cdot |L(K)|}$, *where γ_b is the longest saddle connection in ∂K.*

Note that the conditions depend on the interval $I_{j'}$ and not on the interval I_j.

Proof. We choose

(3) $$c_{M+1} < \epsilon,$$

where ϵ is the constant given by Lemma 1. Then choose

(4) $$c_M^2 < L_0^2 \beta^{2N_0 M} |I_1|,$$

where L_0 is the length of the shortest saddle connection on (X, q).

Then inductively we choose c_i so that

(5) $$c_i^2 (4 + 4\beta^{-N_0 M} + \frac{\beta^{-6-2N_0 M}}{4}(1 + \beta^{N_0 M} + 4\beta^{-N_0 M})) \leq \frac{c_{i+1}^2}{9},$$

The particular choice of constants satisfying these conditions will not be important. Now recall that given j, then j' is determined by (1) and i by (2). Then let $\Omega(j)$ be the set of $c_i \beta^{3+N_0 M}$ shrinkable level i complexes K such that

(A') $|I_{j'}| \cdot L(K) \cdot |\gamma_b| < \beta^{-N_0 M} c_i^2$, where γ_b is the longest boundary saddle connection and
(b') $d(\theta(K), I_{j'}) < \frac{\beta^{N_0 M} c_i^2}{4|\gamma_b| \cdot L(K)}$.

We note that (b') is the opposite of (b) whereas (A') is the opposite of the following strengthening of (a):

(A) $|I_{j'}| \cdot L(K) \cdot |\gamma_b| \geq \beta^{-N_0 M} c_i^2$

Note that $\Omega(j) = \Omega(j')$ since the conditions depend on the interval $I_{j'}$ and not the interval I_j. In the game the combinatorial classes in $\Omega(j)$ will be dealt with one at a time starting at step j', but because they show up at stage j' we wished not to distinguish between them and so we have put the same condition on each of them. We also set the condition

(a') $|I_{j'}| \cdot L(K) \cdot |\gamma_b| < c_i^2$

as the opposite of (a).

We will now describe Alice's strategy in any block of N_0 steps starting with a step j where $j = j'$ or equivalently $j \equiv 1 \bmod N_0$. If $\Omega(j) \neq \emptyset$ choose a combinatorial equivalence class $[K_1]$ of such K and let z_j^1 be the midpoint of the smallest interval

that contains all $\theta(K_1)$ for $K_1 \in \Omega(j)$ in that equivalence class. Alice's strategy is to choose U_j to be an interval of length $\beta|I_j|$ centered at z_j if $d(z_j, \partial I_j) \geq \frac{\beta}{2}|I_j|$ and an interval of the same length that contains z_j and abuts I_j at the endpoint closest to z_j otherwise. If $\Omega(j) = \emptyset$ Alice chooses any subinterval of I_j that has length $\beta|I_j|$.

Next Alice is presented with an interval I_{j+1}. Now consider $\Omega(j) \setminus [K_1]$. If this set is nonempty choose a remaining equivalence class $[K_2]$ and corresponding U_{j+1} of length $\beta|I_{j+1}|$ just as above using the midpoint z_{j+1}^2 of the smallest interval that contains all $\theta(K_2)$ for K_2 in this equivalence class. If it is empty Alice chooses any subinteral of I_{j+1} that has length $\beta|I_{j+1}|$. We continue this a total of N_0 steps ending with I_{j+N_0}.

We shall argue by induction on all j that

(i) for every $c_i\beta^{3+N_0M}$ shrinkable level i complex K, either (a) or (b) holds.
(ii) there are at most N_0 combinatorially distinct $K \in \Omega(j)$.
(iii) for any two combinatorially equivalent $K_1, K_2 \in \Omega(j)$, $|\theta(K_1) - \theta(K_2)| \leq \frac{\beta^{N_0}}{3}|I_{j'}|$.

Note that since the conditions are the same for all j in this block of N_0 indices in some sense this is really an induction on j'.

For $1 \leq j \leq N_0M$, we note that (a) holds for any i-complex K since

(6) $$L(K) \cdot |\gamma_b| \cdot |I_{j'}| \geq L_0^2 \beta^{N_0M}|I_1| > c_M^2 \geq c_i^2.$$

Moreover $\Omega(j) = \emptyset$ since (A') implies

$$L(K) \cdot |\gamma_b| \cdot |I_{j'}| < \beta^{-N_0M}c_i^2 \leq \beta^{-N_0M}c_M^2 < \beta^{N_0M}L_0^2|I_1|.$$

This contradicts the first inequality in (6). This verifies (i)-(iii) for these j.

Now let $j > N_0M$ and suppose that (i)-(iii) have been verified for $n = 1, \ldots, j-1$. We shall prove (i) first, by contradiction. Suppose K is a $c_i\beta^{3+N_0M}$ shrinkable i complex and neither (a) nor (b) holds. Let us write (a_j), (b_j), etc. to indicate the dependence on j. Note that (A_{j-N_0M}) implies (a_j) because $|I_{j'}| \geq \beta^{N_0M}|I_{j'-N_0M}|$. Thus, (A_{j-N_0M}) does not hold, or equivalently, (A'_{j-N_0M}) holds. Note also that (b_{j-N_0M}) implies (b_j) because $I_{j'} \subset I_{j'-N_0M}$. Thus, (b_{j-N_0M}) does not hold, or equivalently, (b'_{j-N_0M}) holds. By the induction hypothesis (i) for $n = j - N_0M$ it follows that (a_{j-N_0M}) holds.

Now set $n = j - N_0M$ and let n' be the corresponding value $n' \equiv 1 \mod N_0$. We have shown $K \in \Omega(n)$. The induction hypothesis (ii) says that there are at most N_0 combinatorially distinct $K \in \Omega(n)$. In Alice's strategy this combinatorial class was considered at some stage m where $n' \leq m \leq n' + N_0 - 1$. Alice chose an interval U_m of size $\beta|I_m|$ centered at the midpoint of the smallest interval that contains all $\theta(K')$ for K' in the same class as K. Now since $m - n' \leq N_0 - 1$, the induction hypothesis (iii) implies that $\theta(K)$ is contained in the middle third of U_m. Therefore

$$d(\theta_K, I_j) \geq \frac{\beta}{3}|I_m| \geq \frac{\beta^{N_0}}{3}|I_{n'}| \geq \frac{\beta^{N_0}c_i^2}{4L(K)|\gamma_b|}.$$

This shows that (b_j) holds for these values of j, contrary to assumption. This verifies (i).

Now we prove (ii), also by contradiction. Suppose $K_1, K_2 \in \Omega(j)$ are combinatorially distinct $c_i\beta^{3+N_0M}$-shrinkable level i-complexes. Without loss of generality

we can assume $L(K_1) \leq L(K_2)$. Assume that they are combinable so that there is $\tilde{\gamma}_2 \in \partial K_2$ that intersects the exterior of K_1. We will arrive at a contradiction.

By the preceding paragraphs, either (a_j) or (b_j) holds for both complexes, but by definition of $\Omega(j)$, (b_j) does not hold. Thus (a_j) and (A_j') hold and therefore, we have

$$(7) \qquad \frac{c_i^2}{L(K_1)|\gamma_b^1|} \leq |I_{j'}| \leq \frac{\beta^{-N_0 M} c_i^2}{L(K_2)|\gamma_b^2|}.$$

which we note implies

$$(8) \qquad L(K_1)|\gamma_b^2| \leq L(K_2)|\gamma_b^2| \leq \beta^{-N_0 M} L(K_1)|\gamma_b^1|.$$

By Lemma 7, there exists a saddle connection σ disjoint from K_1 such that $K' = K_1 \cup \{\sigma\}$ is $c_{i+1}\beta^{3+N_0 M}$-shrinkable, provided K_1 and $\tilde{\gamma}_2$ are jointly $\frac{c_{i+1}\beta^{3+N_0 M}}{6}$-shrinkable. We now check this condition. Using (8), the assumption that $L(K_1) \leq L(K_2)$, and $|\tilde{\gamma}_2| \leq |\gamma_b^2|$, we have $L(K_1)|\gamma_b^1| \geq \beta^{N_0 M} L(K_2)|\gamma_b^2| \geq \beta^{N_0 M} L(K_1)|\tilde{\gamma}_2|$. Together with the assumption that both complexes satisfy (b_j') we have

$$|\theta(K_1) - \theta(K_2)| \leq |I_{j'}| + \frac{\beta^{N_0 M} c_i^2}{4L(K_1)|\gamma_b^1|} + \frac{\beta^{N_0 M} c_i^2}{4L(K_2)|\gamma_b^2|} \leq \frac{\beta^{-N_0 M} c_i^2}{L(K_1)|\tilde{\gamma}_2|} + \frac{c_i^2(1 + \beta^{N_0 M})}{4|\tilde{\gamma}_2| L(K_1)}.$$

Since K_2 is $c_i\beta^{3+N_0 M}$-shrinkable, we have

$$|\theta(K_2) - \theta_{\tilde{\gamma}_2}| \leq \frac{c_i^2 \beta^{6+2N_0 M}}{|\tilde{\gamma}_2| L(K_2)} \leq \frac{c_i^2 \beta^{6+2N_0 M}}{|\tilde{\gamma}_2| L(K_1)}.$$

Then by the triangle inequality,

$$(9) \qquad |\theta(K_1) - \theta_{\tilde{\gamma}_2}| \leq \frac{4c_i^2 \beta^{6+2N_0 M} + c_i^2(1 + \beta^{N_0 M} + 4\beta^{-N_0 M})}{4|\tilde{\gamma}_2| L(K_1)} \leq \frac{\beta^{6+2N_0 M} c_{i+1}^2}{36|\tilde{\gamma}_2| L(K_1)}.$$

The last inequality is a consequence of (5). If $|\tilde{\gamma}_2| \leq L(K_1)$ this says that K_1 and $\tilde{\gamma}_2$ are jointly $\frac{c_{i+1}}{6}\beta^{3+N_0 M}$ shrinkable.

The other case is $L(K_1) \leq |\tilde{\gamma}_2|$. Since K_1 is $c_i\beta^{3+N_0 M}$ shrinkable we have

$$(10) \qquad |\theta(K_1) - \theta_\omega| \leq \frac{c_i^2 \beta^{6+2N_0 M}}{L(K_1)|\omega|}$$

for all $\omega \in K_1$.

Then combining (9) and (10), and the facts that $L(K_1) \geq \beta^{N_0 M}|\tilde{\gamma}_b^2|$ and $L(K_1) \geq |\omega|$ we conclude by the triangle inequality that

$$(11)$$
$$|\theta_{\tilde{\gamma}_b^2} - \theta_\omega| \leq \frac{4c_i^2 \beta^{6+2N_0 M}(1 + \beta^{-N_0 M}) + c_i^2(1 + \beta^{N_0 M} + 4\beta^{-N_0 M})}{4|\tilde{\gamma}_2||\omega|} \leq \frac{c_{i+1}^2 \beta^{6+2N_0 M}}{36|\tilde{\gamma}_2||\omega|}.$$

The last inequality is a restatement of (5). This also says that $\tilde{\gamma}_2$ and K_1 are jointly $\frac{c_{i+1}\beta^{3+N_0 M}}{6}$ shrinkable. By Lemma 7, for some saddle connection σ we can construct an $i + 1$ complex $K' = K \cup \{\sigma\}$ that is $c_{i+1}\beta^{3+N_0 M}$-shrinkable and such that $L(K') \leq 3\max(L(K_1), |\gamma_b'|)$.

If $i = M$, then the choice of $c_{M+1} < \epsilon$ and the fact that there are no level $M + 1$ ϵ complexes says that K' in fact cannot exist and gives the desired contradiction. Again let j' be detrmined by j, and let $j'' = j' - N_0$. For $i < M$, we need to verify that K' satisfies $(a_{j''}')$ and $(b_{j''}')$ to show that K' does not exist for the desired contradiction. Let γ_b' be the longest saddle connection on the boundary of K'.

Since K_2 satisfies (A'_j) we have

$$L(K')|\gamma'_b||I_{j''}| \leq 9\beta^{-N_0}L(K_2)|\gamma_b^2||I_{j'}| < 9\beta^{-N_0M-N_0}c_i^2 < c_{i+1}^2,$$

the last inequality a consequence of (5). Thus, K' satisfies $(a'_{j''})$.

Now

$$d(\theta(K'), I_{j''})$$
$$\leq d(\theta(K'), \theta_{\gamma'_b}) + d(\theta_{\gamma'_b}, \theta(K_1)) + d(\theta(K_1), I_{j''})$$
$$\leq \frac{c_{i+1}^2\beta^{6+2N_0M}}{L(K')|\gamma'_b|} + \frac{4c_i^2\beta^{6+2N_0M}}{L(K_1)|\gamma'_b|} + \frac{\beta^{N_0M}c_i^2}{4L(K_1)|\gamma_b^1|}.$$

The first inequality comes from the fact that K' is $c_{i+1}\beta^{3+N_0M}$ shrinkable. The second inequality comes from the third conclusion of Lemma 7. The last inequality comes from the assumption (b') on K_1 and the fact that $I_{j'} \subset I_{j''}$. However $L(K') \leq 3\max(L(K_1), |\gamma'_b|)$ and

$$|\gamma'_b| \leq \frac{3|\gamma_b^1|}{\beta^{N_0M}} \leq \frac{3L(K_1)}{\beta^{N_0M}},$$

so the above right hand side is at most

$$\frac{c_{i+1}^2\beta^{6+2N_0M}}{L(K')|\gamma'_b|} + \frac{36c_i^2\beta^{6+N_0M}}{L(K')|\gamma'_b|} + \frac{9c_i^2\beta^{-N_0M}}{4L(K')|\gamma'_b|} < \frac{\beta^{N_0M}c_{i+1}^2}{4L(K')|\gamma'_b|}.$$

The last inequality is again a consequence of (5) and the fact that $\beta < \frac{1}{3}$. We have shown $(b'_{j''})$ holds for K'.

This is a contradiction to the induction hypothesis (i) with $j'' = j' - N_0$. We have shown that there cannot be combinatorially distinct K_1, K_2 that are combinable. But then by Lemma 3 there can be at most N_0 different combinatorial complexes in $\Omega(j)$. This verifies (ii).

Now we prove (iii). Since all K are $c_i\beta^{3+N_0M}$-shrinkable

$$|\theta_K - \theta_{\gamma_b}| \leq \frac{c_i^2\beta^{6+2N_0M}}{L(K)|\gamma_b|}$$

so that if K_1, K_2 are combinatorially equivalent hence share a longest boundary saddle γ_b, by (7) and the triangle inequality, we have

$$|\theta(K_1) - \theta(K_2)| \leq \frac{2c_i^2\beta^{6+2N_0M}}{\min(L(K_1), L(K_2))|\gamma_b|} \leq 2\beta^{6+2N_0M}|I_{j'}| \leq \frac{\beta^{N_0}}{3}|I_{j'}|.$$

This proves (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1. By Theorem 2 we are able to ensure that for any level i complex K_i and longest $\gamma_b \in \partial K_i$, we have

$$\max\{|\gamma_b| \cdot L(K) \cdot |I_{j'}|, |\gamma_b| \cdot L(K) \cdot d(\theta(K), I_{j'})\} > \frac{\beta^{N_0M}c_i^2}{4}.$$

In particular this holds when $i = 1$. Since there is only one saddle connection in a 1-complex, and since for any fixed saddle connection γ, $|\gamma|^2|I_{j'}| \to 0$ as $j' \to \infty$, we conclude that for all but finitely many intervals $I_{j'}$ we have

$$|\gamma|^2 d(\theta_\gamma, I_{j'}) = \max\{|\gamma|^2|I_{j'}|, |\gamma|^2 d(\theta_\gamma, I_{j'})\} > \frac{\beta^{N_0M}c_1^2}{4}.$$

Thus if $\phi = \cap_{l=-1}^{\infty} I_{j'}$ is the point we are left with at the end of the game, and γ is a saddle connection, then $|\gamma|^2 |\theta_\gamma - \phi| > \frac{\beta^{N_0 M} c_1^2}{4}$, which by Proposition 1 establishes Theorem 1. $\qquad\square$

References

[1] Kerckhoff, S., Masur, H., Smillie, J *Ergodicity of billiard flows and quadratic differentials*, Annals of Math, 1986 **124** 293-311

[2] Kleinbock, D, Weiss, B: *Bounded geodesics in moduli space.* Int. Math. Res. Not. 2004, **30**, 1551-1560.

[3] McMullen, C. T: *Winning sets, quasiconformal maps and Diophantine approximation.* Geom. Funct. Anal. 2010, **20** 726-740,

[4] Minsky, Y; Weiss, B: *Nondivergence of horocyclic flows on moduli space.* I. Reine Angew. Math. (2002), **552** 131-177

[5] Schmidt, Wolfgang M. *On badly approximable numbers and certain games.* Trans. Amer. Math. Soc. 1966 **123** 178-199.